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Anomalous dielectric relaxation of inhomogeneous media with chaotic structure

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Abstract. The analogue of a Taylor series for fractal functions was derived by treatment of anomalous dielectric relaxation in inhomogeneous media with chaotic structure within the frame of the fractional integro-differential calculus technique. Analysis of a classical problem of polarization of an inhomogeneous medium permitted us to establish the relationship between anomalous relaxation and dimensionality of a temporal fractal ensemble which characterizes a non-equilibrium state of a medium.

1. Introduction

Anomalous non-exponential relaxations have long been and still are a hot topic in the physics of inhomogeneous media [1–24]. Broadly speaking, one may refer to three general relaxation laws which are encountered in the experimental studies of complex systems:

(i) stretched exponential [10–13],

$$f(t) \approx \exp\left[-\left(\frac{t}{\tau}\right)^{\alpha}\right] \qquad 0 < \alpha < 1, t > \tau \tag{1.1}$$

(ii) exponential–logarithmic [14],

$$f(t) \approx \exp\left[-B \cdot \ln^{\alpha}\left(\frac{t}{\tau}\right)\right] \tag{1.2}$$

(iii) algebraic decay [15],

$$f(t) \approx \left(\frac{t}{\tau}\right)^{-\alpha} \tag{1.3}$$

where α, τ and B are the appropriate fitting parameters.

Currently, there seems to be no quantitative microscopic theory for the cited laws [1, 13, 20]; moreover, sometimes even the possibility of such a theory is denied [16]. The main argument is that a spatial inhomogeneity (such as, e.g., a random distribution of impurities within a matrix, or of interatomic spacings in amorphous semiconductors) will necessarily

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result in an extremely board range of microscopic transition rates. Hence, a spatial disorder is expected to induce a temporal energetic disorder.

Another approach to the problem of anomalous relaxations makes use of fractal concepts [3–9, 17–30]. In this case, the problem is analysed using the mathematical language of fractional derivatives [18–30] based on the Riemann–Liouville fractional differentiation operator [31, 32],

$$D^{\alpha}[f(t)] = \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{c}^{t}(t-\tau)^{-\alpha}f(\tau)\,\mathrm{d}\tau \tag{1.4}$$

where $\Gamma(x)$ is the gamma function.

In spite of the reasonable success of the latter approach [3–9, 13–30], use of the fractional derivative as represented by equation (1.4) makes difficult the interpretation of differentiation procedures (for example, the non-zero value of a fractional derivative of a constant), as well as their relevant to the assumed fractal ensemble. One may also note that so far the fractional derivatives have been analysed in essentially phenomenological terms; moreover, the equations based on fractional derivatives were constructed more by intuition (guessed) [3–9], rather than obtained by derivation.

In this context, attempts to construct fractional derivatives and to clarify their relevance to the assumed fractal ensemble are believed to remain feasible for the treatment of the problem of anomalous relaxations.

In this paper the analysis of a classical problem of polarization of an inhomogeneous medium permits us to establish the relationship between anomalous relaxation and dimensionality of a temporal fractal ensemble which characterizes a non-equilibrium state of a medium.

2. Derivative of fractal functions

In general, functions for which the total increment,

$$\Delta_h f(x) = f(x+\Delta x) - f(x) \tag{2.1}$$

can be represented as

$$\Delta_h f = A(\Delta x)^h + \alpha(x)(\Delta x)^h \qquad (\lim \alpha(x) \to 0,\ \text{if } (\Delta x)^h \to 0) \tag{2.2}$$

may be subdivided into two classes.

(i) $h = 1$; $0{:}f(x)$ belongs to the classical ensemble of differentiated functions;

(ii) $h \neq 1$ (Hoelder index): $f(x)$ belongs to the ensemble of functions for which it is not the classical derivative but only the fractional derivative which exists [31, 32],

$$\frac{\mathrm{d}^h f(x)}{\mathrm{d}x^h} = \lim \frac{\Delta_h f}{[\Delta x]^h} \qquad [\Delta x]^h \to 0. \tag{2.3}$$

It can be inferred from equation (2.2) that the increment of the function $\log \Delta^h f$ (in logarithmic metrics) should change linearly with the increment of an independent variable $\log \Delta x$; hence, the standard differential calculus becomes applicable.

It is pertinent to recall here that fractals are defined, sometimes, as continuous functions characterized by absence of derivatives (tangents) at any point, with a curvilinear cone serving as a tangent to the fractal curve trajectory [33]. Apparently, Czech scientist Bolzano was the first to study such continuous, non-differentiated functions around 1830 (the corresponding manuscript was discovered only in 1920 [34]). Wiener's process (i.e., Brownian motion) and Kolmogorov's turbulence (i.e., non-smooth vector field) may be cited as examples of phenomena which can be described by continuous, non-differentiated functions (fractal functions).

The displacement $y(t)$ of a Brownian particle in the former (Wiener) process is defined as [35]

$$|y(t+\Delta t) - y(t)| \approx [\Delta t]^{\alpha} \tag{2.4}$$

whereas the singular velocity of the latter phenomenon (Kolmogorov's turbulent flow) is characterized by [36]

$$\langle|\Delta\vec{\nu}|^{p}\rangle \approx [\Delta x]^{p/3} \tag{2.5}$$

where $\Delta\vec{\nu} = \vec{\nu}(x+\Delta x) - \vec{\nu}(x)$ is the difference of velocities between two point separated by distance Δx.

Assume that a function $f(x)$ is defined on a fractal ensemble Ω_f, of dimensionality $0 < \mathrm{d}_f < 1$, and that a point $x = x_0$ and its vicinity belong to the ensemble Ω_f. It is assumed that $f(x) = 0$ if $x < 0$.

Let us divide a segment $[x, x_0]$ so that the length of each kth fragment at the nth scale level is

$$\Delta x_k^{(n)} = \xi^{n}(x_0 - x) \tag{2.6}$$

where $\xi < 1$ is the scaling factor (i.e., the index of similarity of the ensemble Ω_f).

The number of dividing points of the segment $[x, x_0]$ at the nth step is therefore

$$m_n = 1, 2, \ldots, j^{n+1} \tag{2.7}$$

where j is the number of blocks (i.e., the branching index) involved in the construction of the fractal unit cell ($j = 2$ for the Cantor ensemble).

Let the unit scale at the nth step be $[\Delta x]^{\alpha}$,

$$[\Delta x_k^{(n)}]^{\alpha} = \frac{1}{N_n}(x_0 - x)^{\alpha} \tag{2.8}$$

where $N_1 = j^1, \ldots; N_n = j^n$ (that is, $N_n = j^n$ determines the number of fragments at the nth scale level). This definition of the unit scale for the segment $[x, x_0]$ allows us to associate each point (element) of fractal ensemble with a point of an ultrametric space whose geometrical symbol may be represented by the Cayley tree [37–39].

It follows from equation (2.8) that $\lim_{n\to\infty} x_k^n = 0$; hence, $\Delta x_k^{(n)}$ is an infinitesimal quantity (that is, the ultrametric space becomes continuous at $n \to \infty$). From now on, the increment of the function argument $\Delta x_k^{(n)}$ at the nth step will be denoted as Δx (that is, $\Delta x = \Delta x_k^{(n)}$), while the corresponding coordinates of dividing points will be defined as

$$x_k = x_0 - k\Delta x_k^{(n)} = x_0 - k\Delta x \tag{2.9}$$

where $k = 0, 1, 2, \ldots, j^{n+1}$. Recognition of fractal dimensionality as $d_f = \alpha$ implies, further, $(1/\xi)^{n\alpha} = N_n$ and $\Delta x = (x_0 - x)/(1/\xi)^n$, $[\Delta x]^{\alpha} = [(x_0 - x)^{\alpha}/N_n]$, $(1/\xi)^{n\alpha} = j^n = N_n$.

Consider an increment, $\Delta_\alpha f(x) = f(x_0) - f(x_0 - \Delta x)$; then the kth increment $\Delta_\alpha^k f(x)$ will be determined through binomial coefficients with alternating signs (cf appendix A),

$$\Delta_\alpha^k f(x_0) = \sum_{k=0}^{m}[-1]^k C_n^k (f(x_0 - k\Delta x) \qquad C_m^k = \frac{m!}{k!(m-k)!} \qquad m = j^{n+1} \tag{2.10}$$

and the function $f(x)$ in the vicinity of point x_0 will be

$$f(x) = (1 - \Delta_\alpha)^m f(x_0). \tag{2.11}$$

Using equations (2.6)–(2.11), one can derive an analogue of the Taylor series for function $f(x)$ (cf appendix A):

$$f(x) = \sum_{k=0}^{\infty} a_k (x_0 - x)^{\alpha k} \tag{2.12}$$

where $a_k = (j^k/k!)f^{(\alpha k)}(x_0)$, and $f^{(\alpha k)}(x_0)$ defines the fractional derivative of the kth order of the fractal function $f(x)$ at the point $x = x_0$ as

$$f^{(\alpha k)}(x_0) = \lim_{\Delta x \to 0} \frac{\Delta_\alpha^k f(x_0)}{([\Delta x]^\alpha)^k}. \tag{2.13}$$

The coefficients of the series (2.12) depend both on the fractional derivative of kth order of the fractal function $f(x)$ at the point $x = x_0$ and on the branching index j of the fractal ensemble of which the function $f(x)$ is specified.

It follows from equation (2.13) that the first derivative ($k = 1$) is

$$\frac{\mathrm{d}^\alpha f(x_0)}{\mathrm{d}x^\alpha} = f^{(\alpha)}(x_0) = \lim_{[\Delta x]^\alpha \to 0} \frac{\Delta^\alpha f(x_0)}{[\Delta x]^\alpha} = \lim_{[\Delta x]^\alpha \to 0} \frac{f(x_0) - f(x_0 - \Delta x)}{[\Delta x]^\alpha}; \tag{2.14}$$

thus, equation (2.3) is recovered.

In a similar way, one can also specify the integral of function $f(x)$ on fractal ensemble Ω_f as a limit of integral summation (cf appendix B),

$$\int_a^b f(x)[\mathrm{d}x]^\alpha = \lim_{(n\to\infty)} \sum_{k=1}^{\infty} f(x_0 - (k-1)\Delta x)[\Delta x]^\alpha. \tag{2.15}$$

For convenience in the further use of the fractional derivative, let us introduce the differentiation operator D^α,

$$D^\alpha f(x) = \frac{\mathrm{d}^\alpha f(x)}{[\mathrm{d}(x-a)]^\alpha} = \lim_{\Delta x \to 0} \frac{f(x) - f(x - \Delta x)}{[\Delta x]^\alpha} \qquad 0 < \alpha \leqslant 1. \tag{2.16}$$

The integration operator I^α will be defined as

$$I^\alpha f(x) = \int_{-\infty}^{x} f(x)[\mathrm{d}x]^\alpha \tag{2.17}$$

so that $I^\alpha = D^{-\alpha}$, i.e.,

$$I^\alpha f(x) = D^{-\alpha} f(x) = \int_{-\infty}^{x} f(x)[\mathrm{d}x]^\alpha. \tag{2.18}$$

Consider the function

$$\Phi(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} (x-t)^{\alpha-1} f(t)\,\mathrm{d}t \tag{2.19}$$

It can be shown that

$$D^\alpha \Phi(x) = f(x); \tag{2.20}$$

hence,

$$D^{-\alpha} f(x) = \int_{-\infty}^{x} f(x)[\mathrm{d}x]^\alpha = \Phi(x). \tag{2.21}$$

Thus,

$$D^{1-\alpha} f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\Phi(x) \tag{2.22}$$

or

$$D^\beta f(x) = \frac{1}{\Gamma(\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \int_{-\infty}^{x} (x-t)^{-\beta} f(t)\,\mathrm{d}t \tag{2.23}$$

where $\alpha + \beta = 1$, $0 < \alpha \leqslant 1$ and $f(x) = 0$, if $x < 0$.

Summarizing, the developed fractional–integral concepts establish the link with the procedure of construction of the fractal ensemble which determines the function $f(x)$.

3. Dielectric relaxation

The potential of fractional derivatives will become evident, and the relationship between the exponent α in equation (1.1)–(1.13) and the fractal dimensionality d_f will be established, in the subsequent treatment of the classical problem of polarization $P(t)$ of a dielectric medium (which is, in fact, equivalent to the general problem of relaxation of internal parameters of a non-equilibrium phase).

Assume that $P(t)$ contains two contributions [1, 40],

$$P(t) = P_0 + P_1(t) \tag{3.1}$$

where the former one ($P_0 = \chi_0 E$) varies exactly (at least, with negligibly small retardation) as the applied field E, while the latter time-dependent one, $P_1(t)$, is retarded. Let $P^* = \chi_\infty E$ be the upper limit (at fixed E); then the instantaneous rate of approach of the contribution $P_1(t)$ to this limit will be higher, the larger the amplitude $(\chi_\infty E - P_1(t))$. Hence, the corresponding relaxation equation may be written as [39]

$$\frac{\mathrm{d}P_1(t)}{\mathrm{d}t} = \frac{1}{\tau}(\chi_\infty E - P_1(t)) \tag{3.2}$$

where τ is the relaxation time. After integration, one derives from equation (3.2):

$$P(t) = P_0 + P_1(t) = \left[\chi_0 + \chi_\infty\left(1 - \exp\left(-\frac{t}{\tau}\right)\right)\right]E \tag{3.3}$$

(for the field fixed at E), and

$$P(\omega) = P_0 + P_1(\omega) = [\chi_0 + \chi_\infty/(1 + \mathrm{i}\omega\tau)]E \tag{3.4}$$

(for the field alternating as $E = E_0\,\mathrm{e}^{\mathrm{i}\omega t}$).

Therefore, the dielectric permittivity of a medium may be defined, finally, as [40]

$$\varepsilon = \varepsilon_\infty + \frac{\varepsilon_0 - \varepsilon_\infty}{1 + \mathrm{i}\omega\tau} \tag{3.5}$$

where

$$\varepsilon_\infty = \lim_{\omega\to\infty} \varepsilon \qquad \varepsilon_0 = \varepsilon|_{\omega=0}.$$

Let us consider now the non-equilibrium state of a fractal-like medium assuming that this non-equilibrium state is characterized by many events such that each next event is separated by a certain time interval τ_i from a previous event. In this case, some intervals will be eliminated from a continuous process of system evolution by a definite law. Assume that such a process is caused by a temporal fractal state of dimensionality d_f; the corresponding relaxation equation can be written as

$$D^\alpha P_1(t) = \frac{1}{\tau}(\chi_\infty E - P_1(t)) \tag{3.6}$$

and rearranged as

$$[1 + \tau D^\alpha]P_1(t) = \chi_\infty E \tag{3.7}$$

where $\alpha = d_f$.

The latter equation (3.7) can be solved using the Laplace transform (cf appendix C),

$$[1 + (\tau p)^\alpha]\overline{P_1}(p) = \frac{\chi_\infty E}{p} \tag{3.8}$$

which yields

$$\overline{P_1}(p) = \frac{\chi_\infty E}{p}\frac{1}{1 + (\tau p)^\alpha}. \tag{3.9}$$

In so far as

$$\frac{1}{1+(p\tau)^{\alpha}} = \frac{(p\tau)^{-\alpha}}{1+(p\tau)^{-\alpha}} = \sum_{n=0}^{\infty}(-1)^n (p\tau)^{-\alpha(n+1)} \tag{3.10}$$

the solution of equation (3.9) in the domain of originals will have the following form:

$$P_1(t) = \chi_\infty E \sum_{n=0}^{\infty} \frac{(-1)^n (t/\tau)^{\alpha(n+1)}}{\Gamma[\alpha(n+1)+1]} \tag{3.11}$$

where $\Gamma(x)$ is the gamma-function. Therefore,

$$P(t) = P_0 + P_1(t) = \left[\chi_0 + \chi_\infty \sum_{n=0}^{\infty} \frac{(-1)^n (t/\tau)^{\alpha(n+1)}}{\Gamma[\alpha(n+1)+1]}\right] E. \tag{3.12}$$

After substitution of $\alpha = 1$ into equation (3.12), equation (3.3) may be recovered; in fact,

$$P(t) = \left[\chi_0 + \chi_\infty \sum_{n=0}^{\infty} \frac{(-1)^n (t/\tau)^{(n+1)}}{\Gamma(n+2)}\right] E = \left[\chi_0 + \chi_\infty \left(1 - \exp\left(-\frac{t}{\tau}\right)\right)\right] E \tag{3.13}$$

(in the derivation, the standard equation (3.14) was used):

$$\sum_{n=0}^{\infty} \frac{(-1)^n (z)^n}{\Gamma[(n+1)]} = \exp(-z) \qquad z = \frac{t}{\tau}. \tag{3.14}$$

Thus, the cross-over from a strictly exponential to an anomalous relaxation pattern can be associated with change of a continuous distribution of relaxation times ($\alpha = 1$) into a fractal-like one ($0 < \alpha = d_f < 1$).

If follows from equation (3.12) that

$$P(t) \approx \left[\left(1 - \exp\left(-\left(\frac{t}{\tau}\right)^{\alpha}\right)\right)\right] E \tag{3.15}$$

which can be compared with equation (1.1)–(1.3).

In the case of alternating field, the Fourier transform of equation (3.7) yields (cf appendix C)

$$P(\omega) = \lfloor \chi_0 + \chi_\infty/(1+\mathrm{i}\omega\tau)^{\alpha} \rfloor E \tag{3.16}$$

and the dielectric permittivity will be

$$\varepsilon = \varepsilon_\infty + \frac{\varepsilon_0 - \varepsilon_\infty}{1+(\mathrm{i}\omega\tau)^{\alpha}}. \tag{3.17}$$

The real $\mathrm{Re}\,\varepsilon(\omega)$ and imaginary $\mathrm{Im}\,\varepsilon(\omega)$ parts of the total dielectric permittivity in equation (3.17) are, respectively,

$$\mathrm{Re}\,\varepsilon(\omega) = \varepsilon_0 \left[\gamma + \frac{(1-\gamma)[1+(\omega\tau)^{\alpha}\cos(\pi\alpha/2)]}{1+2(\omega\tau)^{\alpha}\cos(\pi\alpha/2)+(\omega\tau)^{2\alpha}}\right] \tag{3.18}$$

$$\mathrm{Im}\,\varepsilon(\omega) = \varepsilon_0 \left[\frac{(\gamma-1)[1+(\omega\tau)^{\alpha}\sin(\pi\alpha/2)]}{1+2(\omega\tau)^{\alpha}\cos(\pi\alpha/2)+(\omega\tau)^{2\alpha}}\right] \tag{3.19}$$

therefore, the dielectric loss tangent will be

$$\tan\delta = (\gamma - 1)\left[\frac{(\omega\tau)^{\alpha}}{1+2(\omega\tau)^{\alpha}\cos(\pi\alpha/2)+(\omega\tau)^{2\alpha}}\right] \tag{3.20}$$

where $\gamma = \varepsilon_\infty/\varepsilon_0$.

Equations (3.18)–(3.20), respectively, were used to construct the plots of the real, $\mathrm{Re}\,\varepsilon(\omega)/\varepsilon_0$ (figure 1), and of the imaginary, $\mathrm{Im}\,\varepsilon(\omega)/\varepsilon_0$ (figure 2), parts of complex dielectric permittivity, as well as of the dielectric loss tangent $\tan\delta$ (figure 3) as a function of $\log\omega\tau$ for a medium with $\gamma = \varepsilon_\infty/\varepsilon_\infty = 10$. As can be easily verified, the relaxation spectrum pattern strongly depends on the dimensionality of the temporal fractal ensemble $\alpha = d_f$.

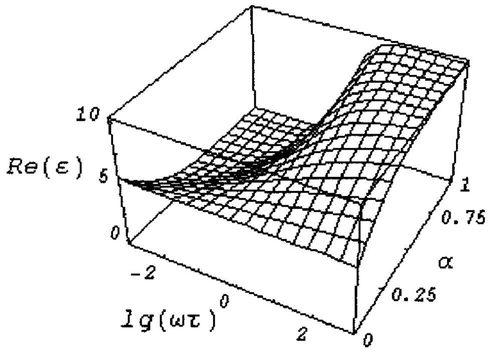

Figure 1. Disperse dependence Re $\varepsilon/\varepsilon_0$ at the different values of parameter α.

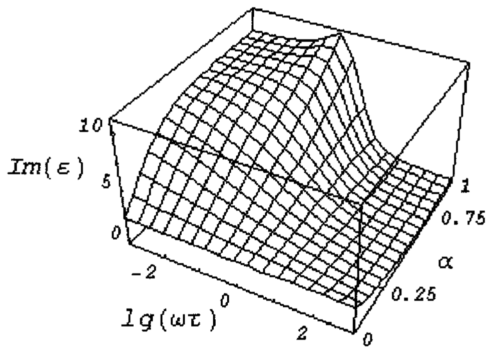

Figure 2. Disperse dependence Im $\varepsilon/\varepsilon_0$ at the different values of parameter α.

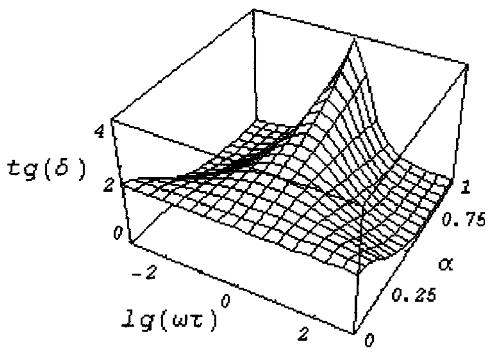

Figure 3. Disperse dependence tan δ at the different values of parameter α.

4. Conclusions

The potential of the fractional derivative technique is demonstrated on the example of dielectric relaxation in a non-homogeneous medium. This approach allows for a simple and transparent analysis of the dependence of the pattern of an anomalous relaxation spectrum on the dimensionality of temporal fractal ensemble.

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Appendix A. Analogue of Taylor series for a fractal function

Consider the increment,

$$\Delta_\alpha f(x) - f(x_0) - f(x_0 - \Delta x); \tag{A.1}$$

hereafter, this increment will be called the first difference. Hence, the second difference $\Delta_\alpha^2 f(x)$ will be defined as the squared operator Δ_α,

$$\begin{aligned}\Delta_\alpha^2 f(x) &= \Delta_\alpha(\Delta_\alpha f(x)) = \Delta_\alpha f(x_0) - \Delta_\alpha f(x_0 - \Delta x)\\ &= f(x_0) - 2f(x_0 - \Delta x) + f(x_0 - 2\Delta x).\end{aligned} \tag{A.2}$$

In a similar fashion, the third difference will be

$$\Delta_\alpha^3 f(x) = f(x_0) - 3f(x_0 - \Delta x) + 3f(x_0 - 2\Delta x) - f(x_0 + 3\Delta x).$$

Thus, the kth difference $\Delta_\alpha^k f(x)$ will be determined through the binomial coefficients with alternating signs,

$$\Delta_\alpha^k f(x_0) = \sum_{k=0}^{m}[-1]^k C_n^k f(x_0 - k\Delta x) \qquad C_m^k = \frac{m!}{k!(m-k)!} \qquad m = j^{n+1}. \tag{A.3}$$

As follows from the definition of Δ_α,

$$f(x_0 - \Delta x) = f(x_0) - \Delta_\alpha f(x_0) = (1 - \Delta_\alpha) f(x_0) \tag{A.4}$$

where 1 is defined as a symbol of an identical operator. Therefore, one can write

$$f(x - 2\Delta x) = (1 - \Delta_\alpha) f(x_0 - \Delta x) = (1 - \Delta_\alpha)^2 f(x_0). \tag{A.5}$$

In a general case,

$$f(x_0 - k\Delta x) = (1 - \Delta_\alpha)^k f(x_0) \tag{A.6}$$

therefore,

$$f(x) = (1 - \Delta_\alpha)^m f(x_0) \tag{A.7}$$

in so far as, according to equation (2.6),

$$x = x_0 - m\Delta x \qquad \text{where } m = j^{n+1}.$$

Using the binomial expansion for $(1 - \Delta_\alpha)^k$, one obtains

$$f(x) = \sum_{k=0}^{m}[-1]^k C_m^k \Delta_\alpha^k f(x_0). \tag{A.8}$$

Transform the common term in the r.h.s. of equation (A.8) as

$$\begin{aligned}C_m^k \Delta_\alpha^k f(x_0) &= C_m^k \frac{\Delta_\alpha^k f(x_0)}{([\Delta x]^\alpha)^k}([\Delta x]^\alpha)^k = \frac{m(m-1)\ldots(m-k+1)}{k!}\frac{\Delta_\alpha^k f(x_0)}{([\Delta x]^\alpha)^k}\frac{(x_0 - x)^{\alpha k}}{N_n^k}\\ &= P_{mk}\frac{\Delta_\alpha^k f(x_0)}{k!([\Delta x]^\alpha)^k}\end{aligned} \tag{A.9}$$

where

$$P_{mk} = \frac{m(m-1)\ldots(m-k+1)}{N_n^k} \qquad k = 1, 2, \ldots, m. \tag{A.10}$$

Hence, equation (A.8) can be rewritten as

$$f(x) = \sum_{k=0}^{m} [-1]^k \frac{P_{mk}}{k!} \frac{\Delta_\alpha^k f(x_0)}{([\Delta x]^\alpha)^k} (x_0 - x)^{\alpha k}. \tag{A.11}$$

Assuming finite k and infinite m ($m \to \infty$), one arrives at the analogue of Taylor's series for function $f(x)$,

$$f(x) = \sum_{k=0}^{\infty} \frac{j^k f^{(\alpha k)}(x_0)}{k!} (x_0 - x)^{\alpha k}$$

or

$$f(x) = \sum_{k=0}^{\infty} a_k (x_0 - x)^{\alpha k} \tag{A.12}$$

where $\alpha_k = (j^k/k!) f^{(\alpha k)}(x_0)$ and $f^{\alpha k}(x_0)$ is specified as

$$f^{\alpha k}(x_0) = \lim_{n\to\infty} \frac{\Delta_\alpha^k f(x_0)}{([\Delta x]^\alpha)^k}. \tag{A.13}$$

Appendix B. Fractional integral

Consider and integral sum on the segment $[a, b]$ belonging to Ω_f,

$$\begin{aligned} \textstyle\sum_f &= [f(x_0) + f(x_0 - \Delta x) + \cdots + f(x_0 - (n-1)\Delta x)][\Delta x]^\alpha \\ &= \sum_{k=1}^{n} f(x_0 - (k-1)\Delta x)[\Delta x]^\alpha \qquad b = x_0 - (n-1)\Delta x \end{aligned} \tag{B.1}$$

where $[\Delta x]^\alpha$ is defined by equation (2.6). The integral will correspond to the limit of the integral sum (B.1),

$$\lim_{\substack{n\to\infty \\ (\Delta l x^\alpha \to 0)}} \textstyle\sum_f = I \tag{B.2}$$

that is,

$$\int_a^b f(x)\,[\mathrm{dx}]^\alpha = \lim_{\substack{\mathrm{n}\to\infty \\ (\Delta\mathrm{x}\to 0)}} \textstyle\sum_{\mathrm{f}} \tag{B.3}$$

or

$$\int_a^b f(x)[\mathrm{d}x]^\alpha = \lim_{\substack{n\to\infty \\ (\Delta x\to 0)}} \sum_{k=1}^{\infty} f(x_0 - (k-1)\Delta x)[\Delta x]^\alpha. \tag{B.4}$$

Appendix C. Fourier and Laplace transforms

The Fourier transform is defined as

$$\bar{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x \tag{C.1}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{f}(k)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k. \tag{C.2}$$

In the case of a fractional derivative, the Fourier transform will be

$$\overline{\frac{\mathrm{d}^\alpha f(x)}{[\mathrm{d}x]^\alpha}} = (\mathrm{i}k)^\alpha \bar{f}(k) \tag{C.3}$$

where

$$\frac{\mathrm{d}^\alpha f(x)}{[\mathrm{d}x]^\alpha} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mathrm{i}k)^\alpha \bar{f}(k)\, \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}k \tag{C.4}$$

The Laplace transform is defined as

$$\bar{f}(p) = \int_0^\infty f(x) \exp(-px)\, \mathrm{d}x \tag{C.5}$$

with the original function $f(x)$

$$f(x) = \frac{1}{2\pi\mathrm{i}} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} \bar{f}(p) \exp(px)\, \mathrm{d}p \qquad \mathrm{Re}\, p \geqslant a. \tag{C.6}$$

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